

Distributed Interference Cancellation in Multiple Access Channels

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Abstract

In this paper, we consider a Gaussian multiple access channel with multiple independent additive white Gaussian interferences. Each interference is known to exactly one transmitter non-causally. The capacity region is characterized to within a constant gap regardless of channel parameters. These results are based on a layered modulo-lattice scheme which realizes distributed interference cancellation.

Index Terms

Dirty paper coding, dirty multiple access channels, distributed interference cancellation, modulo-lattice scheme, binary expansion model.

I. INTRODUCTION

In modern wireless communication systems, interference has become the major barrier for efficient utilization of available spectrum. In many scenarios, interferences are originated from sources close to transmitters and hence can be inferred by intelligent transmitters, while receivers cannot due to physical limitations. With the knowledge of interferences as side information, transmitters are able to encode their information against interferences and mitigate them, even though receivers cannot distinguish interferences from desired signals. The simplest information

theoretic model for studying such interference mitigation is the single-user point-to-point dirty-paper channel [1], which is a special case of state-dependent memoryless channels with the state¹ known non-causally to the transmitter [2]. It is shown that the effect of interference can be completely removed in the additive white Gaussian noise (AWGN) channel when the interference is also additive white Gaussian [1]. As for multi-user scenarios, it has been found that when perfect state information (the additive interference) is available non-causally at all transmitters, the capacity region of the AWGN multiple access channel (MAC) is not affected by the additive white Gaussian interference [3] [4]. When the state information is known *partially* to different transmitters in the MAC, however, the capacity loss caused by the interference is unbounded as the signal-to-noise ratios increase [5] [6]. Since each transmitter only has partial knowledge about the interference, interference cancellation has to be realized in a *distributed* manner.

In this paper, we consider a K -user Gaussian MAC with K independent additive white Gaussian interferences. Each interference is known to exactly one transmitter non-causally. Transmitter i , for all $i = 1, \dots, K$, aims to deliver a message w_i to the receiver reliably through the channel depicted in Fig. 1, where

$$y = \sum_{i=1}^K x_i + \sum_{i=1}^K s_i + z,$$

and $z \sim \mathcal{N}(0, N_o)$ is the AWGN noise. Interference $s_i \sim \mathcal{N}(0, Q_i)$, $i = 1, \dots, K$, independent of everything else, is known non-causally to transmitter i *only*. Power constraint at transmitter i is P_i , $i = 1, \dots, K$. Define channel parameters $\text{SNR}_i := P_i/N_o$, $\text{INR}_i := Q_i/N_o$, for $i = 1, 2$. User i 's rate is denoted by R_i , $i = 1, \dots, K$. Throughout this paper, without loss of generality we assume that $P_1 \geq P_2 \geq \dots \geq P_K$.

A. Related Works

State-dependent networks with partial state knowledge available at different nodes have been studied in various scenarios. Kotagiri *et al.*[7] study the state-dependent two-user MAC with state non-causally known to a transmitter, and for the Gaussian case they characterize the capacity asymptotically at infinite interference ($K = 2, Q_1 = \infty, Q_2 = 0$) as the informed transmitter's power grows to infinity. Somekh-Baruch *et al.*[6] study the problem with the same set-up as [7]

¹In dirty-paper channel, the state is the additive interference.

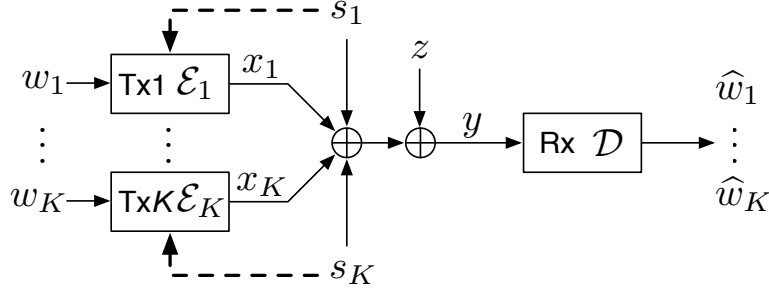


Fig. 1. Channel Model

while the informed transmitter knows the other's message, and they characterize the capacity region completely. Zaidi *et al.*[8] study another case of degraded message set. The achievability part of [7], [6], and [8] are based on random binning. Philosof *et al.*[5], on the other hand, characterize the capacity region of the doubly-dirty MAC to within a constant gap at infinite interference (i.e., $K = 2$, $Q_1 = Q_2 = \infty$), by lattice strategies [9]. They also show that strategies based on Gaussian random binning is unboundedly worse than lattice-based strategies. Zaidi *et al.*[10] [11] and Akhbari *et al.*[12] study a state-dependent relay channel where the state is only known either at the source or the relay.

B. Main Contribution

We characterize the capacity region of the channel in Fig. 1 to within $K \log_2 K$ bits, regardless of channel parameters P_i 's, Q_i 's, and N_o . The constant gap only depends on the number of users in the channel and is independent of channel parameters, providing a strong guarantee on the performance for any fixed K . Our approach to this problem is first investigating a *binary expansion model* of the original channel. The binary expansion model is a natural extension of the linear deterministic model proposed in [13] to the case with additive interferences known to transmitters. After characterizing the capacity region of the binary expansion model, we then make use of the intuitions and techniques developed there to derive outer bounds and build up achievability results for the original Gaussian problem. Such approach has been successfully applied to various problems in network information theory, including [14], [15], [16], [17], [18], etc.

For the achievability part we propose a layered modulo-lattice scheme consisting of K layers,

based on the intuition drawn from the study of the binary expansion model. Layer i is shared among user $1, \dots, i$, and the hierarchy of the layers is $1 \rightarrow 2 \rightarrow \dots \rightarrow K$, from the top to the bottom. Each layer treats the signals sent at higher layers as *interference*, each of which is known non-causally to exactly one transmitter. In each layer $i \in 1, 2, \dots, K$, we use a modulo-lattice scheme to realize distributed interference cancellation, which is a simpler version of the single layer scheme in [5]. For the converse part, we first extend the ideas in [19] to derive matching outer bounds for the binary expansion model and then use the same technique to prove bounds in the Gaussian scenario.

C. Notations

Notations used in this paper are summarized below:

- Throughout the paper, the block coding length is denoted by N . A sequence of random variables $x[1], \dots, x[N]$ is denoted by x^N and boldface \mathbf{x} interchangeably.
- Logarithms are of base 2 if not specified. We use short-hand notations $(\cdot)^+$ to denote $\max\{0, \cdot\}$ and $\log^+(\cdot)$ to denote $(\log(\cdot))^+$.
- We use the short-hand notation $[k_1 : k_2]$ to denote a set/tuple (k_1, \dots, k_2) and $v_{[k_1:k_2]}$ to denote $(v_{k_1}, \dots, v_{k_2})$ if $k_1 \leq k_2$, respectively. If $k_1 > k_2$, $[k_1 : k_2]$ and $v_{[k_1:k_2]}$ denote the empty set ϕ .
- Similarly, for a set of indices S , we use v_S to denote the collection $\{v_i \mid i \in S\}$.

D. Paper Organization

The rest of this paper is organized as follows. In Section II, we first introduce and formalize the binary expansion model, which serves as an auxiliary channel for the original one. Then we characterize the capacity region of the auxiliary channel and draw important intuitions for solving the original problem. In Section III, we propose the layered modulo-lattice scheme and derive its achievable rates. Then we show that the achievable rate region is within a constant gap to the proposed outer bounds in Section IV. Finally, we conclude the paper in Section V.

II. A BINARY EXPANSION MODEL FOR GAUSSIAN MAC WITH ADDITIVE INTERFERENCES

To approach the distributed interference cancellation problem in Gaussian multiple access channels (MAC), we first study a binary expansion model of the original problem. Solutions to

the original problem can be inferred by solving the auxiliary problem in this model. The model is a natural generalization of the linear deterministic model proposed in [13], with random states acting as additive interferences. We formally define the model as follows.

Definition 2.1 (Binary Expansion Model): The binary expansion MAC with additive interferences known to transmitters, corresponding to the original Gaussian problem, is defined by nonnegative integers

$$n_i := \left\lfloor \frac{1}{2} \log^+ \text{SNR}_i \right\rfloor, \quad m_i := \left\lfloor \frac{1}{2} \log^+ \text{INR}_i \right\rfloor, \quad i = 1, \dots, K, \quad (1)$$

transmitted signals $x_{b,i} \in \mathbb{F}_2^q$, interferences $s_{b,i} \in \mathbb{F}_2^q$ for $i \in [1 : K]$, and received signal

$$y_b = \sum_{i=1}^K A^{q-n_i} x_{b,i} + \sum_{i=1}^K A^{q-m_i} s_{b,i}, \quad (2)$$

where additions are modulo-two component-wise, $q = \max \{n_i, m_i : i \in [1 : K]\}$, and $A \in \mathbb{F}_2^{q \times q}$ is the shift matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \quad (3)$$

Each interference $s_{b,i}$ consists of q i.i.d. Bernoulli $(\frac{1}{2})$ bits and is known to transmitter i , for $i \in [1 : K]$.

Here we use subscript b to draw distinction from the original channel model. Note that the condition $P_1 \geq P_2 \geq \dots \geq P_K$ implies $n_1 \geq n_2 \geq \dots \geq n_K$.

An example is depicted in Fig. 2, where $n_1 = 4, n_2 = 2, m_1 = 5, m_2 = 3$.

The main result in this section is the characterization of capacity region of the auxiliary channel, summarized in the following theorem and two lemmas. To distinguish notations from the original Gaussian problem, lower-cases letters are used to represent rates in the binary expansion model.

Lemma 2.2 (Outer Bounds): If $r_{[1:K]} \geq 0$ is achievable, it satisfies the following: for all $k \in [1 : K]$,

$$\sum_{i=k}^K r_i \leq \bar{r}_k(n_{[k:K]}, m_{[k+1:K]}; K), \quad (4)$$

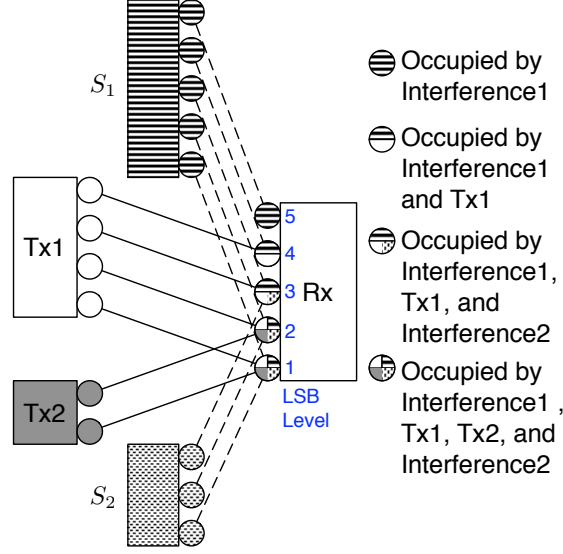


Fig. 2. The Binary Expansion Model. The numbering in blue denotes the ordering of the least significant bit (LSB) levels.

where

$$\bar{r}_k(n_{[k:K]}, m_{[k+1:K]}; K) := \max \{m_{[k+1:K]}, n_k\} - \sum_{i=k+1}^K (m_i - \max \{m_{[i+1:K]}, n_i\})^+. \quad (5)$$

Proof: The proof is detailed in Section II-C. ■

Lemma 2.3 (Inner Bounds): If $r_{[1:K]} \geq 0$ satisfies the following: for all $k \in [1 : K]$,

$$\sum_{i=k}^K r_i \leq \underline{r}_k(n_{[k:K]}, m_{[k+1:K]}; K) \quad (6)$$

it is achievable. Here

$$\underline{r}_k(n_{[k:K]}, m_{[k+1:K]}; K) := \sum_{i=k}^K (n_i - \max \{m_{[i+1:K]}, n_{i+1}\})^+. \quad (7)$$

Proof: The proof is detailed in Section II-B. ■

Theorem 2.4 (Capacity of the Binary Expansion Model): $r_{[1:K]} \geq 0$ is achievable, if and only if it satisfies the following: for all $k \in [1 : K]$,

$$\sum_{i=k}^K r_i \leq \underline{r}_k(n_{[k:K]}, m_{[k+1:K]}; K), \quad (8)$$

where $\underline{r}_k(n_{[k:K]}, m_{[k+1:K]}; K) = \bar{r}_k(n_{[k:K]}, m_{[k+1:K]}; K) = \underline{r}_k(n_{[k:K]}, m_{[k+1:K]}; K)$.

Proof: To show $\bar{r}_k(n_{[k:K]}, m_{[k+1:K]}; K) = \underline{r}_k(n_{[k:K]}, m_{[k+1:K]}; K)$ for all $k \in [1 : K]$, we shall use induction backwards.

1) $k = K$: $\bar{r}_K(n_K; K) = n_K = \underline{r}_K(n_K; K)$.

2) Suppose the claim is correct for $k = l$. For $k = l - 1$,

$$\bar{r}_{l-1}(n_{[l-1:K]}, m_{[l:K]}; K) - \bar{r}_l(n_{[l:K]}, m_{[l+1:K]}; K) \quad (9)$$

$$= \max\{m_{[l:K]}, n_{l-1}\} - \max\{m_{[l+1:K]}, n_l\} - (m_l - \max\{m_{[l+1:K]}, n_l\})^+ \quad (10)$$

$$= \max\{m_{[l:K]}, n_{l-1}\} - \max\{m_{[l+1:K]}, n_l, m_l\} \quad (11)$$

$$= \max\{m_{[l:K]}, n_l, n_{l-1}\} - \max\{m_{[l:K]}, n_l\} \quad (12)$$

$$= \max\{\max\{m_{[l:K]}, n_l\}, n_{l-1}\} - \max\{m_{[l:K]}, n_l\} \quad (13)$$

$$= (n_{l-1} - \max\{m_{[l:K]}, n_l\})^+ = \underline{r}_{l-1}(n_{[l-1:K]}, m_{[l:K]}; K) - \underline{r}_l(n_{[l:K]}, m_{[l+1:K]}; K). \quad (14)$$

Hence, $\bar{r}_{l-1}(n_{[l-1:K]}, m_{[l:K]}; K) = \underline{r}_{l-1}(n_{[l-1:K]}, m_{[l:K]}; K)$. By induction principle, the proof is complete. \blacksquare

A. Motivating Examples

Before we formally prove the converse and the achievability, we first give a couple of examples to illustrate the high-level intuition behind the result. Such intuitions not only work for the binary expansion model, but also carry over to the original Gaussian setting. For simplicity, all examples are two-user ($K = 2$), with fixed $(n_1, n_2) = (4, 2)$ and various (m_1, m_2) . They are depicted in Fig. 3. Although the total number of bit levels of y_b is $q = \max\{n_1, n_2, m_1, m_2\}$, referring to Fig. 2 only the first $\max\{n_1, n_2\} = 4$ least significant bit (LSB) levels are those can be potentially used for communicating information, since none of the transmitters can access the upper bit levels owing to power constraints. Therefore, with the side information of interferences at transmitters, they try to cancel interferences in these 4 levels as much as possible.

The first example (Fig. 3(a)) illustrates the situation where $m_1 \leq n_1$ and $m_2 \leq n_2$. Transmitter 1 can completely cancel interference $s_{b,1}$ since it only occupies $m_1 = 3$ LSB levels of y_b . Transmitter 2 can also cancel interference $s_{b,2}$ completely since it only occupies $m_2 = 1$ LSB level of y_b . Therefore, all bit levels are free from interference, and the capacity region is $r_1 + r_2 \leq 4, r_2 \leq 2$ which is the same as the clean MAC.

The second example (Fig. 3(b)) illustrates the situation where $m_1 \geq n_1$ and $m_2 \leq n_2$. Transmitter 1 cannot completely cancel interference $s_{b,1}$ since it occupies $m_1 = 5$ LSB levels

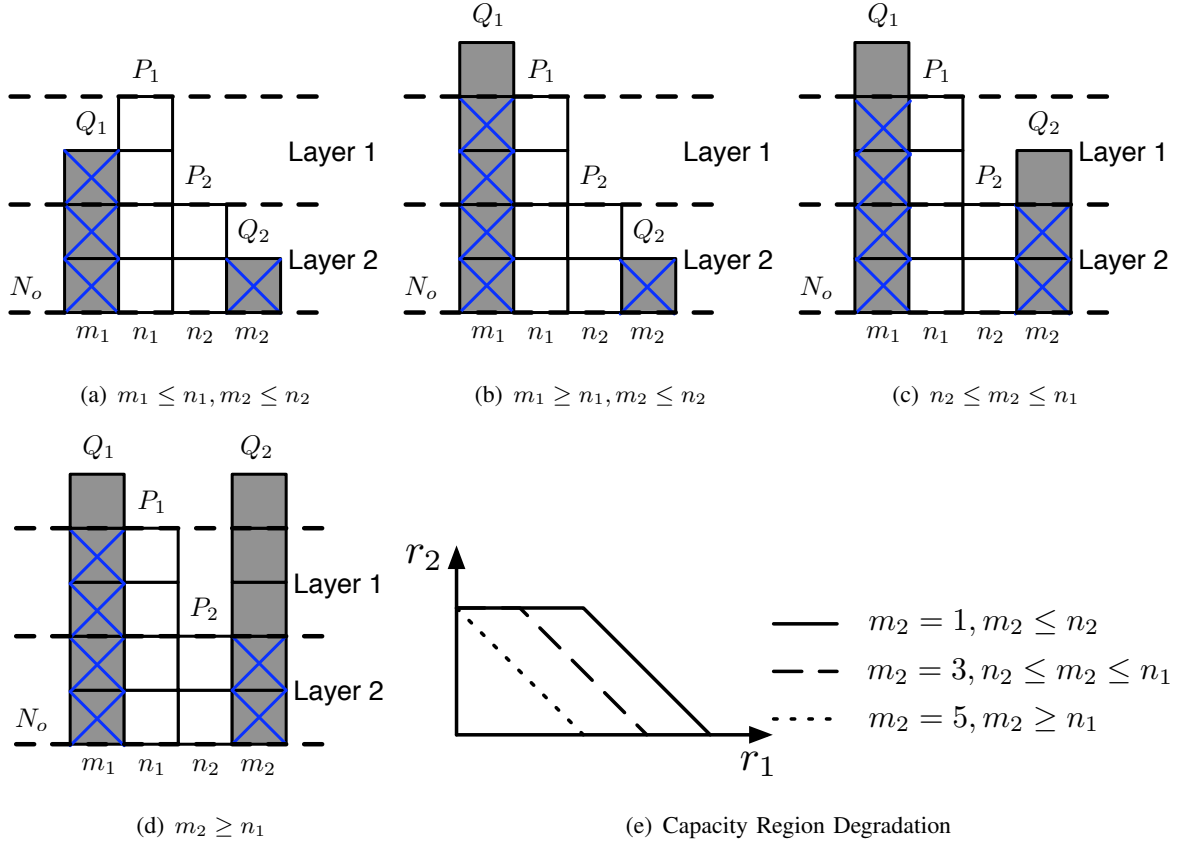


Fig. 3. Examples. Note that by definition, $n_i \leftrightarrow P_i$, $m_i \leftrightarrow Q_i$, and the bottom line corresponds to the noise level. Blue crosses denote that interference bits (shaded) can be cancelled.

of y_b , while transmitter 1 has access to only $n_1 = 4$ LSB levels. However, it can cancel those in the first 4 LSB levels. Transmitter 2 can again cancel interference $s_{b,2}$ completely. Therefore, all 4 LSB levels are free from interference, and the capacity region is $r_1 + r_2 \leq 4, r_2 \leq 2$ which is again the same as the clean MAC.

From the above examples, we see that the strength of interference $s_{b,1}$ does not effect the capacity region, since the only bit levels that matter are the $\max\{n_1, n_2\} = n_1 = 4$ LSB levels, and transmitter 1 can always “clean up” the interference caused by $s_{b,1}$ in these bit levels. On the other hand, the strength of interference $s_{b,2}$ *does* affect the capacity region, as discussed below.

The third example (Fig. 3(c)) illustrates the situation where $n_2 \leq m_2 \leq n_1$. Since transmitter 2 only has access to $n_2 = 2$ LSB levels, it cannot cancel the interference caused by $s_{b,2}$ at the third LSB level. Therefore, the level is no longer useful and cannot be used by transmitter 1. The fourth LSB level, however, is clean after transmitter 1’s interference cancellation. The capacity

region becomes $r_1 + r_2 \leq 3, r_2 \leq 2$.

The last example (Fig. 3(d)) illustrates the situation where $m_2 \geq n_1$. Again transmitter 2 cannot do anything about $s_{b,2}$ except at the 2 LSB levels. Therefore, the third and the fourth bit levels are both corrupted and cannot be used. The capacity region becomes $r_1 + r_2 \leq 2$.

Fig. 3(e) depicts the degradation of the capacity regions due to various strengths of $s_{b,2}$. From the above discussions, we make the following observations.

- 1) The strength of the interference that is known to the strongest transmitter, that is, $s_{b,1}$, does not affect the capacity region, as in the single-user point-to-point case.
- 2) Based on the interference cancellation capability of each transmitter (its transmit power), the bit levels of y_b can be partitioned into K layers (here $K = 2$): layer 1, consisting of the third and the fourth LSB levels, and layer 2, consisting of the first and second levels. In the bottom layer 2, both interferences caused by $s_{b,1}$ and $s_{b,2}$ can be completely cancelled. In this layer both users share n_2 bit levels. On the other hand, in the top layer 1, only the interference caused by $s_{b,1}$ can be cancelled, while that caused by $s_{b,2}$ cannot. Hence in this layer user 1 can only use $(n_1 - m_2)^+$ levels.

These observations lead to a natural way for establishing achievability, which is detailed in Section II-B. For the converse, the above discussion gives the intuitive explanation why the lack of knowledge about $s_{b,2}$ at transmitter 1 degrades the capacity region. In Section II-C we give a formal converse proof.

B. Achievability

Each transmitter, say i , cancels the interference it knows, $s_{b,i}$, as much as it can. If $m_i \leq n_i$, then $s_{b,i}$ can be completely canceled. If $m_i > n_i$, then the top most $(m_i - n_i)$ levels of $s_{b,i}$ cannot be removed, and the bit levels of y_b occupied by this chunk can never be used to convey data by any user. Since the channel is linear and the interferences are additive, the effect of interference cancellation remains for other users.

Superimposed upon interference cancellation, the scheme consists of K layers. Layer i is from the $(n_{i+1} + 1)$ -th level of LSB to the n_i -th level at the receiver, $i \in [1 : K]$. In layer i , user $[1 : i]$ can transmit. Therefore, we have the following achievable rates in layer i , $i \in [1 : K]$:

$r_{[1:i]}^{(i)} \geq 0$ satisfying

$$\sum_{l=1}^i r_l^{(i)} \leq (n_i - \max \{m_{[i+1:K]}, n_{i+1}\})^+. \quad (15)$$

User i 's rate is the aggregate of its rates from layer i to layer K : $r_i = \sum_{l=i}^K r_i^{(l)}$. Apply Fourier-Motzkin elimination we establish Lemma 2.3.

C. Converse Proof

Next we prove the outer bounds in Lemma 2.2.

Proof: Let

$$y_{b,k} := \sum_{i=k}^K \underbrace{A^{q-n_i} x_{b,i}}_{x_i} + \sum_{i=k}^K \underbrace{A^{q-m_i} s_{b,i}}_{s_i}. \quad (16)$$

Here we use x_i to denote $A^{q-n_i} x_{b,i}$ and s_i to denote $A^{q-m_i} s_{b,i}$ for notational convenience. It is easy to distinguish these notations from those in the original Gaussian model based on the context.

If $r_{[1:K]}$ is achievable, for any $k \in [1 : K]$ by Fano's inequality and data processing inequality, we have

$$N \left(\sum_{i=k}^K r_i - \epsilon_N \right) \quad (17)$$

$$\leq I(w_{[k:K]}; y_b^N | w_{[1:k-1]}) \quad (18)$$

$$\stackrel{(a)}{\leq} I(w_{[k:K]}; y_b^N | w_{[1:k-1]}, s_{[1:k-1]}^N) \quad (19)$$

$$= H(y_b^N | w_{[1:k-1]}, s_{[1:k-1]}^N) - H(y_b^N | w_{[1:K]}, s_{[1:k-1]}^N) \quad (20)$$

$$= H(y_{b,k}^N | w_{[1:k-1]}, s_{[1:k-1]}^N) - H(y_{b,k}^N | w_{[1:K]}, s_{[1:k-1]}^N) \quad (21)$$

$$\stackrel{(b)}{=} I(w_{[k:K]}; y_{b,k}^N) = I(w_{[k:K]}, s_{[k:K]}^N; y_{b,k}^N) - I(s_{[k:K]}^N; y_{b,k}^N | w_{[k:K]}) \quad (22)$$

$$\stackrel{(c)}{=} H(y_{b,k}^N) - \sum_{i=k}^K H(s_i^N) + H(s_{[k:K]}^N | y_{b,k}^N, w_{[k:K]}) \quad (23)$$

$$= H(y_{b,k}^N) - \sum_{i=k}^K H(s_i^N) + \sum_{i=k}^K H(s_i^N | y_{b,k}^N, w_{[k:K]}, s_{[k:i-1]}^N) \quad (24)$$

$$\stackrel{(d)}{\leq} H(y_{b,k}^N) - H(s_k^N) + H(s_k^N | y_{b,k}^N) - \sum_{i=k+1}^K H(s_i^N) + \sum_{i=k+1}^K H(s_i^N | y_{b,i}^N, w_{[i:K]}) \quad (25)$$

$$\stackrel{(e)}{\leq} H(y_{b,k}^N | s_k^N) - \sum_{i=k+1}^K H(s_i^N) + \sum_{i=k+1}^K \min \left\{ H(s_i^N), H\left(x_i^N + \sum_{l=i+1}^K (x_l^N + s_l^N)\right) \right\} \quad (26)$$

$$\leq N \left\{ \max \{m_{[k+1:K]}, n_k\} - \sum_{i=k+1}^K m_i + \sum_{i=k+1}^K \min \{m_i, \max \{m_{[i+1:K]}, n_i\}\} \right\}, \quad (27)$$

where $\epsilon_N \rightarrow 0$ as $N \rightarrow \infty$. (a) is due to the facts that conditioning reduces entropy and that $s_{[1:k-1]}^N$ is independent of $w_{[k:K]}$. (b) is due to the fact that $(w_{[k:K]}, s_{[k:K]}^N, y_{b,k}^N)$ and $(w_{[1:k-1]}, s_{[1:k-1]}^N)$ are independent. (c) is due to the fact that $\{w_{[k:K]}, s_{[k:K]}^N\}$ are mutually independent and $y_{b,k}^N$ is a function of $\{w_{[k:K]}, s_{[k:K]}^N\}$. (d) is due to conditioning reduces entropy and the fact that $(w_{[i:K]}, s_{[i:K]}^N, y_{b,i}^N)$ and $(w_{[k:i-1]}, s_{[k:i-1]}^N)$ are independent. (e) is due to the fact that $y_{b,i}^N = x_i^N + s_i^N + \sum_{l=i+1}^K (x_l^N + s_l^N)$.

It is straightforward to see that (27) = $N\bar{r}_k(n_{[k:K]}, m_{[k+1:K]}; K)$. Proof complete. \blacksquare

D. Implication on the Gaussian Problem

By investigating the binary expansion model, we gain intuitions about how to solve the original Gaussian problem. For the outer bounds, we will mimic the proof in Section II-C. For the achievability in the binary expansion model, interference cancellation is realized by simply subtracting interferences from the transmit signals. Due to linearity of the channel and the fact that there is no interaction among different bit levels, if an interference, say, a component of $s_{b,1}$, is cancelled by transmitter 1, it will remain cancelled for other users as well. To realize such distributed interference cancellation in the Gaussian scenario, however, Philosof *et al.*[5] show that Gelfand-Pinsker scheme based on Gaussian random binning is not sufficient. Instead, they propose a modulo-lattice scheme which can carry out this task. Motivated by the layered nature in the achievability of the binary expansion model, we propose a *layered* modulo-lattice scheme, generalized from the single-layer scheme in [5], to realize distributed interference cancellation in all layers, and show that it achieves the capacity region to within a constant number of bits.

III. LAYERED MODULO-LATTICE SCHEME

In this section we first give a brief review on lattices and propose the modulo-lattice scheme used in each layer of our layered architecture. Then we connect all layers, describe the overall architecture, and derive the achievable rates in all layers.

A. A Primer on Lattices

Before introducing the modulo-lattice scheme, first we give some basic definitions and facts about lattices. For more detailed introduction, please refer to [5] and the references therein. For completeness, the following basic and useful facts adapted from [5] are introduced.

An N -dimensional lattice Λ is defined as

$$\Lambda := \{\mathbf{l} = B\mathbf{i} : \mathbf{i} \in \mathbb{Z}^N\}, \quad (28)$$

where $B \in \mathbb{R}^{N \times N}$ is non-singular. By definition, the origin $\mathbf{0} \in \Lambda$.

A natural procedure associated to lattice Λ is to quantize points in \mathbb{R}^N to the nearest lattice point. The nearest neighbor quantizer associated with lattice Λ is defined as

$$Q_\Lambda(\mathbf{x}) := \arg \min_{\mathbf{l} \in \Lambda} \|\mathbf{x} - \mathbf{l}\|, \quad \forall \mathbf{x} \in \mathbb{R}^N. \quad (29)$$

Here $\|\cdot\|$ denote the Euclidean norm.

Another natural procedure is to take the modulo on a lattice. For any $\mathbf{x} \in \mathbb{R}^N$, its modulo on lattice Λ is the “quantization error”

$$\mathbf{x} \bmod \Lambda := \mathbf{x} - Q_\Lambda(\mathbf{x}). \quad (30)$$

Note that the modulo-lattice operation satisfies the distributive property: for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$,

$$[(\mathbf{x} \bmod \Lambda) + \mathbf{y}] \bmod \Lambda = [\mathbf{x} + \mathbf{y}] \bmod \Lambda. \quad (31)$$

The basic Voronoi region of lattice Λ is defined as

$$\mathcal{V} := \{\mathbf{x} \in \mathbb{R}^N : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}. \quad (32)$$

We denote the volume of \mathcal{V} by V , $V = \int_{\mathcal{V}} d\mathbf{x}$.

The second moment of the a lattice Λ is defined by the second moment per dimension of a uniform distribution over the basic Voronoi region \mathcal{V} :

$$\sigma_\Lambda^2 := \frac{1}{N} \frac{\int_{\mathcal{V}} \|\mathbf{x}\|^2 d\mathbf{x}}{V}. \quad (33)$$

The normalized second moment is defined by

$$G(\Lambda) := \frac{\sigma_\Lambda^2}{V^{2/N}}. \quad (34)$$

Note that the normalized second moment of a lattice is always lower bounded by $\frac{1}{2\pi e}$ [20].

The following lemmas [20] turn out to be useful for computing achievable rates.

Lemma 3.1: For a given N -dimensional lattice Λ with basic Voronoi region \mathcal{V} , if random vector $\mathbf{Y} \sim \text{Unif}(\mathcal{V})$, then

$$h(\mathbf{Y}) = \log(V) = \frac{N}{2} \log\left(\frac{\sigma_\Lambda^2}{G(\Lambda)}\right). \quad (35)$$

Lemma 3.2: Consider an N -dimensional lattice Λ has the minimal normalized second moment. If random vector $\mathbf{Y} \sim \text{Unif}(\mathcal{V})$, then its covariance matrix is white: $K_{\mathbf{Y}} = \sigma_\Lambda^2 I_N$. Moreover, there exists a sequence of such lattices $\{\Lambda_N, N \in \mathbb{N}\}$, that is *good for quantization*, in the sense that they attain the lower bound $\frac{1}{2\pi e}$ as $N \rightarrow \infty$:

$$\lim_{N \rightarrow \infty} G(\Lambda_N) = \frac{1}{2\pi e}. \quad (36)$$

B. Modulo-Lattice Scheme in Each Layer

In each layer, we shall use the following canonical modulo-lattice scheme, which is a simplified version of that in [5].

Consider a generic layer k where the subset of participating users is $S^{(k)} \subseteq [1 : K]$. The received signal can be written as

$$\mathbf{y} = \sum_{i \in S^{(k)}} \mathbf{x}_i^{(k)} + \sum_{i \in S^{(k)}} \mathbf{s}_i^{(k)} + \mathbf{z}^{(k)}, \quad (37)$$

where $\mathbf{x}_i^{(k)}$ denotes user i 's transmit signal in this layer, $\mathbf{s}_i^{(k)}$ denotes the *interference* in this layer that is known to user i , and $\mathbf{z}^{(k)}$ denotes the effective aggregate *noise* in this layer. All the transmit signals, interferences, and the noise are mutually independent. The difference between interference and noise is that, interference is mitigated using side information precoding, while noise cannot and hence persists in the received signal. As we shall see in the overall architecture of our layered strategy, interferences $\mathbf{s}_i^{(k)}$ and effective noise $\mathbf{z}^{(k)}$ will contain the signals sent in other layers, and hence is not necessary Gaussian.

The canonical modulo-lattice scheme is configured by three parameters: (1) an N -dimensional lattice $\Lambda^{(k)}$, (2) its second moment $\Theta^{(k)}$, and (3) $S^{(k)} \subseteq [1 : K]$, the subset of users participating in the transmission. For each user $i \in S^{(k)}$, its corresponding sub-encoder in this layer uses lattice $\Lambda^{(k)}$ with second moment $\Theta^{(k)}$ and basic Voronoi region $\mathcal{V}^{(k)}$ to modulate its sub-message $w_i^{(k)}$ in

this layer. Its codeword, $\mathbf{v}_i^{(k)}$, is generated according to $\text{Unif}(\mathcal{V}^{(k)})$ with rate $R_i^{(k)}$. The transmit signal $\mathbf{x}_i^{(k)}$ is generated according to the following modulo-lattice operation:

$$\mathbf{x}_i^{(k)} = \left[\mathbf{v}_i^{(k)} - \alpha^{(k)} \mathbf{s}_i^{(k)} - \mathbf{d}_i^{(k)} \right] \bmod \Lambda^{(k)}, \quad (38)$$

where $\mathbf{d}_i^{(k)} \sim \text{Unif}(\mathcal{V}^{(k)})$ independent of everything else, is the dither known at the receiver (common randomness).

The corresponding decoder in this layer, upon receiving \mathbf{y} , first multiplies \mathbf{y} by $\alpha^{(k)}$, adds the dithers back, and then takes the modulo $\Lambda^{(k)}$ operation. The output becomes

$$\mathbf{y}^{(k)} = \left[\alpha^{(k)} \mathbf{y} + \sum_{i \in S^{(k)}} \mathbf{d}_i^{(k)} \right] \bmod \Lambda^{(k)} \quad (39)$$

$$= \left[\mathbf{y} - (1 - \alpha^{(k)}) \mathbf{y} + \sum_{i \in S^{(k)}} \mathbf{d}_i^{(k)} \right] \bmod \Lambda^{(k)} \quad (40)$$

$$= \left[\sum_{i \in S^{(k)}} \left[\mathbf{v}_i^{(k)} - \alpha^{(k)} \mathbf{s}_i^{(k)} - \mathbf{d}_i^{(k)} \right] \bmod \Lambda^{(k)} + \sum_{i \in S^{(k)}} \mathbf{s}_i^{(k)} + \mathbf{z}^{(k)} \right. \\ \left. - (1 - \alpha^{(k)}) \left(\sum_{i \in S^{(k)}} \mathbf{x}_i^{(k)} + \sum_{i \in S^{(k)}} \mathbf{s}_i^{(k)} + \mathbf{z}^{(k)} \right) + \sum_{i \in S^{(k)}} \mathbf{d}_i^{(k)} \right] \bmod \Lambda^{(k)} \quad (41)$$

$$= \left[\sum_{i \in S^{(k)}} \mathbf{v}_i^{(k)} + \alpha^{(k)} \mathbf{z}^{(k)} - (1 - \alpha^{(k)}) \sum_{i \in S^{(k)}} \mathbf{x}_i^{(k)} \right] \bmod \Lambda^{(k)} \quad (42)$$

$$= \left[\sum_{i \in S^{(k)}} \mathbf{v}_i^{(k)} + \mathbf{z}_{\text{eq}}^{(k)} \right] \bmod \Lambda^{(k)}, \quad (43)$$

where $\mathbf{z}_{\text{eq}}^{(k)} := \alpha^{(k)} \mathbf{z}^{(k)} - (1 - \alpha^{(k)}) \sum_{i \in S^{(k)}} \mathbf{x}_i^{(k)}$ denotes the *effective noise* in the *effective modulo-lattice channel* in layer k . From the first line, due to dithers the output signal $\mathbf{y}^{(k)} \sim \text{Unif}(\mathcal{V}^{(k)})$. Moreover, $\mathbf{z}_{\text{eq}}^{(k)}$ is independent of $\mathbf{v}_{S^{(k)}}$ due to dithering [5].

C. Overall Architecture

Now we are ready to describe the overall architecture of our layered modulo-lattice scheme.

Encoding

For encoding, we shall use an inductive way to describe from the top layer 1 to the bottom layer K , which corresponds to the order of encoding.

1) Layer 1: In this layer, the set of participating users is $S^{(1)} = \{1\}$. We choose the modulation lattice $\Lambda^{(1)}$ to be the one that attaining the minimal normalized second moment with second

moment $\Theta^{(1)} = P_1 - P_2$. The interference $\mathbf{s}_1^{(1)} = \mathbf{s}_1$. The sub-encoder $\mathcal{E}_1^{(1)}$ generates $\mathbf{x}_1^{(1)}$ based on (38) for $k = 1$, and feeds $\mathbf{s}_1^{(2)} := \mathbf{s}_1^{(1)} + \mathbf{x}_1^{(1)}$ to the next-layer sub-encoder $\mathcal{E}_1^{(2)}$.

2) Layer $k, 1 < k < K$: The set of participating users is $S^{(k)} = [1 : k]$. The modulation lattice $\Lambda^{(k)}$ is the one that attaining the minimal normalized second moment with second moment $\Theta^{(k)} = P_k - P_{k+1}$. The known interference $\mathbf{s}_i^{(k)} = \mathbf{s}_i^{(k-1)} + \mathbf{x}_i^{(k-1)}$, for all $i \in [1 : k-1]$, and $\mathbf{s}_k^{(k)} = \mathbf{s}_k$. The sub-encoder $\mathcal{E}_i^{(k)}$ generates $\mathbf{x}_i^{(k)}$ based on (38), and feeds $\mathbf{s}_i^{(k+1)} := \mathbf{s}_i^{(k)} + \mathbf{x}_i^{(k)}$ to the next-layer sub-encoder $\mathcal{E}_i^{(k+1)}$, for all $i \in S^{(k)} = [1 : k]$.

3) Layer K : The set of participating users is $S^{(K)} = [1 : K]$. The modulation lattice $\Lambda^{(K)}$ is the one that attaining the minimal normalized second moment with second moment $\Theta^{(K)} = P_K$. The known interference $\mathbf{s}_i^{(K)} = \mathbf{s}_i^{(K-1)} + \mathbf{x}_i^{(K-1)}$, for all $i \in [1 : K-1]$, and $\mathbf{s}_K^{(K)} = \mathbf{s}_K$. The sub-encoder $\mathcal{E}_i^{(K)}$ generates $\mathbf{x}_i^{(K)}$ based on (38) for $k = K$.

Decoding

The receiver decodes layer $k \in [1 : K]$ with sub-decoder $\mathcal{D}^{(k)}$. Unlike the sequential operation at the sub-encoders, these sub-decoders work *in parallel*. $\mathcal{D}^{(k)}$ takes the received signal \mathbf{y} as input, which can be written as (37), and takes the operation in (39) – (43) to generate $\mathbf{y}^{(k)}$. With the above-mentioned encoding operations, the effective noise

$$\mathbf{z}^{(k)} = \begin{cases} \mathbf{z} + \sum_{l=k+1}^K \left(\mathbf{s}_l + \sum_{i=1}^l \mathbf{x}_i^{(l)} \right), & 1 \leq k \leq K-1 \\ \mathbf{z}, & k = K \end{cases} \quad (44)$$

$$\text{Cov} [\mathbf{z}^{(k)}] = \begin{cases} \left(N_o + \sum_{l=k+1}^K (Q_l + l\Theta^{(l)}) \right) I_N, & 1 \leq k \leq K-1 \\ N_o I_N, & k = K \end{cases} := N^{(k)} I_N, \quad (45)$$

due to our choice of lattices and Lemma 3.2. $N^{(k)}$ denotes the effective per-symbol noise variance in layer k . Due to dithering, indeed $\mathbf{x}_{[1:k]}^{(k)}$, $\mathbf{s}_{[1:k]}^{(k)}$, and $\mathbf{z}^{(k)}$ are mutually independent. Based on $\mathbf{y}^{(k)}$, it performs joint typicality decoding as in standard MAC to find $\mathbf{v}_{S^{(k)}}^{(k)}$, where $S^{(k)} = [1 : k]$.

The overall architecture of transmitters and receiver is depicted in Fig. 4.

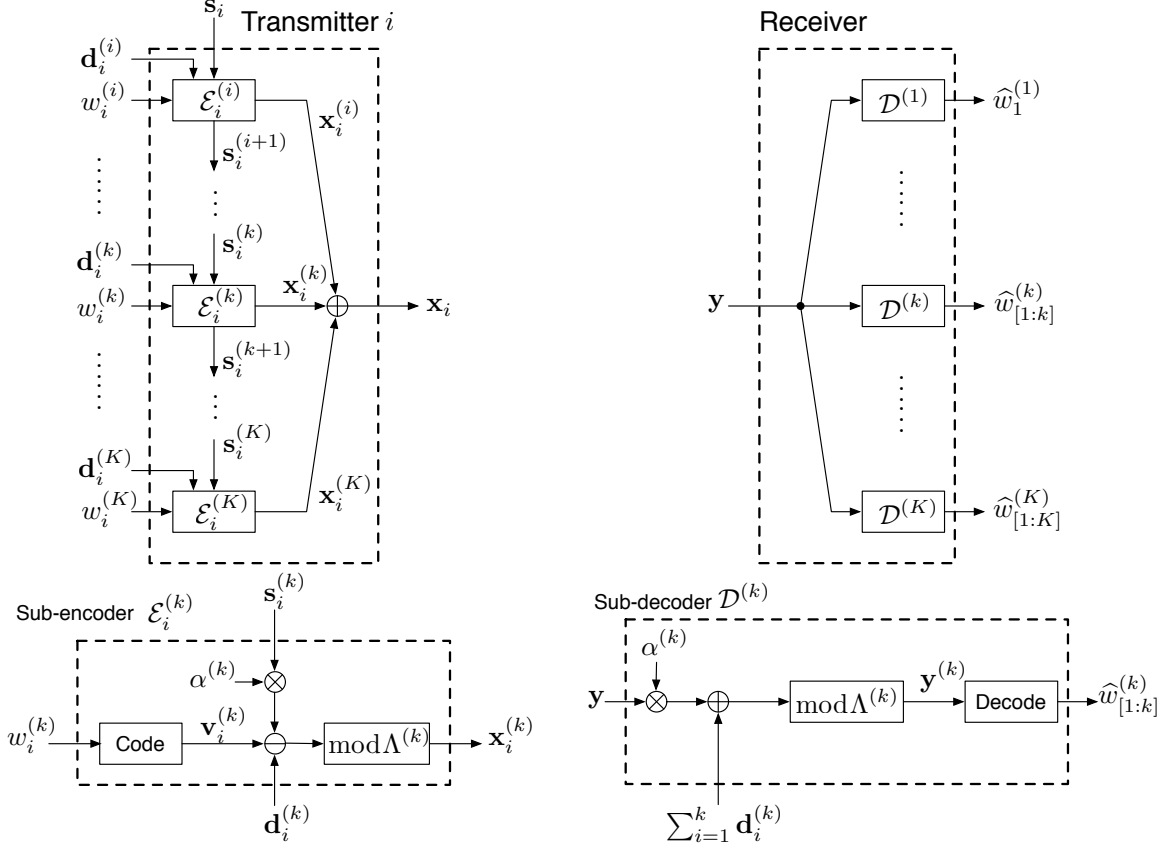


Fig. 4. Transmitter and Receiver Architecture

D. Achievable Rates in Each Layer

The achievable rates of the scheme in this layer can be derived following the same line of analysis as in [5]: non-negative rate tuples $R_{[1:k]}^{(k)}$ is achievable, if

$$N \sum_{i=1}^k R_i^{(k)} \leq I(\mathbf{v}_{[1:k]}^{(k)}; \mathbf{y}^{(k)}) = h(\mathbf{y}^{(k)}) - h\left(\left[\sum_{i=1}^k \mathbf{v}_i^{(k)} + \mathbf{z}_{\text{eq}}^{(k)}\right] \bmod \Lambda^{(k)} \middle| \mathbf{v}_{[1:k]}^{(k)}\right). \quad (46)$$

Since $\mathbf{y}^{(k)} \sim \text{Unif}(\mathcal{V}^{(k)})$, due to Lemma 3.1, the first term $h(\mathbf{y}^{(k)}) = \frac{N}{2} \log\left(\frac{\Theta^{(k)}}{G(\Lambda^{(k)})}\right)$. Moreover, since the modulo operation only reduces the entropy, the second term can be upper bounded as follows:

$$h\left(\left[\sum_{i=1}^k \mathbf{v}_i^{(k)} + \mathbf{z}_{\text{eq}}^{(k)}\right] \bmod \Lambda^{(k)} \middle| \mathbf{v}_{[1:k]}^{(k)}\right) \quad (47)$$

$$\leq h\left(\sum_{i=1}^k \mathbf{v}_i^{(k)} + \mathbf{z}_{\text{eq}}^{(k)} \middle| \mathbf{v}_{[1:k]}^{(k)}\right) \quad (48)$$

$$\stackrel{(a)}{=} h(\mathbf{z}_{\text{eq}}^{(k)}) \quad (49)$$

$$\stackrel{(b)}{\leq} \frac{N}{2} \log \left(2\pi e \left((\alpha^{(k)})^2 N^{(k)} + (1 - \alpha^{(k)})^2 k \Theta^{(k)} \right) \right). \quad (50)$$

(a) is due to the fact that $\mathbf{z}_{\text{eq}}^{(k)}$ and $\mathbf{v}_{[1:k]}^{(k)}$ are independent. (b) is due to the fact that Gaussian distribution is the entropy maximizer for a given covariance matrix, and that the covariance matrix of $\mathbf{z}_{\text{eq}}^{(k)} = \alpha^{(k)} \mathbf{z}^{(k)} - (1 - \alpha^{(k)}) \sum_{i=1}^k \mathbf{x}_i^{(k)}$ is

$$\text{Cov}[\mathbf{z}_{\text{eq}}^{(k)}] = (\alpha^{(k)})^2 N^{(k)} I_N + (1 - \alpha^{(k)})^2 k \Theta^{(k)} I_N, \quad (51)$$

based on (45) and Lemma 3.2.

Hence, combining the above two, we obtain a lower bound on the right-hand side of (46):

$$\frac{N}{2} \log \left(\frac{\Theta^{(k)}}{G(\Lambda^{(k)})} \right) - \frac{N}{2} \log \left(2\pi e \left((\alpha^{(k)})^2 N^{(k)} + (1 - \alpha^{(k)})^2 k \Theta^{(k)} \right) \right) \quad (52)$$

$$= N \left\{ \frac{1}{2} \log \left(\frac{\Theta^{(k)}}{(\alpha^{(k)})^2 N^{(k)} + (1 - \alpha^{(k)})^2 k \Theta^{(k)}} \right) - \frac{1}{2} \log (2\pi e G(\Lambda^{(k)})) \right\}. \quad (53)$$

Based on Lemma 3.2, there exists a sequence of lattices satisfying (36), and therefore all non-negative rates satisfying

$$\sum_{i=1}^k R_i^{(k)} \leq \frac{1}{2} \log^+ \left(\frac{\Theta^{(k)}}{(\alpha^{(k)})^2 N^{(k)} + (1 - \alpha^{(k)})^2 k \Theta^{(k)}} \right) \quad (54)$$

are achievable in layer k , $k \in [1 : K]$. Note that the optimal choice of $\alpha^{(k)}$ is the MMSE coefficient $\alpha^{(k)} = \frac{(N^{(k)})(k\Theta^{(k)})}{N^{(k)} + k\Theta^{(k)}}$, and the resulting rate constraint is

$$\sum_{i=1}^k R_i^{(k)} \leq \frac{1}{2} \log^+ \left(\frac{1}{k} + \frac{\Theta^{(k)}}{N^{(k)}} \right) \quad (55)$$

$$= \frac{1}{2} \log^+ \left(\frac{1}{k} + \frac{P_k - P_{k+1}}{N_o + \sum_{l=k+1}^K Q_l + \sum_{l=k+1}^{K-1} l(P_l - P_{l+1}) + K P_K} \right) \quad (56)$$

$$= \frac{1}{2} \log^+ \left(\frac{1}{k} + \frac{P_k - P_{k+1}}{N_o + (k+1)P_{k+1} + Q_{k+1} + \sum_{j=k+2}^K (P_j + Q_j)} \right) \quad (57)$$

$$= \frac{1}{2} \log^+ \left(\frac{N_o + kP_k + \sum_{j=k+1}^K (P_j + Q_j)}{k \left(N_o + kP_{k+1} + \sum_{j=k+1}^K (P_j + Q_j) \right)} \right). \quad (58)$$

For notational convenience, we denote $P_{K+1} = \text{SNR}_{K+1} = 0$.

In the next section, we derive outer bounds based on similar proof techniques as in the binary expansion model (Section II-C), derive inner bounds based on the discussion above, and show that they are within a constant number of bits to one another.

IV. CONSTANT GAP TO CAPACITY

The main result is summarized in the following lemmas and theorem.

Lemma 4.1 (Outer Bounds): If $R_{[1:K]} \geq 0$ is achievable, it satisfies the following: for all $k \in [1 : K]$,

$$\sum_{i=k}^K R_i \leq \bar{R}_k (\text{SNR}_{[k:K]}, \text{INR}_{[k+1:K]}; K), \quad (59)$$

where

$$\bar{R}_k (\text{SNR}_{[k:K]}, \text{INR}_{[k+1:K]}; K) := \left\{ \begin{array}{l} \frac{1}{2} \log \left(1 + \sum_{i=k+1}^K 2 (\text{SNR}_i + \text{INR}_i) + \text{SNR}_k \right) \\ - \sum_{i=k+1}^K \frac{1}{2} \log^+ \left(\frac{\text{INR}_i}{1 + \sum_{l=i+1}^K 2 (\text{SNR}_l + \text{INR}_l) + \text{SNR}_i} \right) \end{array} \right\} \quad (60)$$

Proof: The technique is similar to the converse proof for the binary expansion model. See Appendix A for detail. ■

Lemma 4.2 (Inner Bounds): If $R_{[1:K]} \geq 0$ satisfies the following: for all $k \in [1 : K]$,

$$\sum_{i=k}^K R_i \leq \underline{R}_k (\text{SNR}_{[k:K]}, \text{INR}_{[k+1:K]}; K) \quad (61)$$

it is achievable. Here

$$\underline{R}_k (\text{SNR}_{[k:K]}, \text{INR}_{[k+1:K]}; K) := \sum_{i=k}^K \frac{1}{2} \log^+ \left(\frac{1 + i \text{SNR}_i + \sum_{j=i+1}^K (\text{SNR}_j + \text{INR}_j)}{i (1 + i \text{SNR}_{i+1} + \sum_{j=i+1}^K (\text{SNR}_j + \text{INR}_j))} \right) \quad (62)$$

Proof: Based on Section III-D, user i 's aggregate rate R_i is the sum of rates in all layers in which it participates, that is, layer i to layer K : $R_i = \sum_{l=i}^K R_i^{(l)}$. Applying Fourier-Motzkin elimination, we complete the proof. ■

Theorem 4.3 (Constant Gap to Capacity):

The above inner and outer bounds are within $(K - k + 1) (\log K + \frac{1}{2})$ bits for user k , for all $k \in [1 : K]$.

Proof: See Appendix B. ■

Remark 4.4: An alternative way to show the inner and outer bounds are within a constant is using the binary expansion model as an interface. Under the conversion in Definition 2.1, it turns out that the outer bounds in Lemma 2.2 and Lemma 4.1 are within a constant number of bits, as well as the inner bounds in Lemma 2.3 and Lemma 4.2. Then by Theorem 2.4, which shows that the inner and outer bounds match in the binary expansion model, it is immediate to establish the constant-gap-to-optimality result in the Gaussian scenario. Moreover, it justifies the usage of the binary expansion model in solving this problem, in the sense that its capacity region uniformly approximate that of the original Gaussian model.

V. CONCLUSION

Costa's landmark paper [1] demonstrates that with proper precoding, in the point-to-point AWGN channel the effect of additive interference can be mitigated as if there were no interference, as long as the interference is known to the transmitter non-causally. In the multi-user scenario, however, when the interference is known partially to each node in the network, such conclusion no longer holds. Moreover, in the two-user doubly-dirty MAC, Philosof *et al.* [5] shows that a natural extension of Costa's Gaussian random binning scheme performs unboundedly worse than a lattice-based strategy.

In this paper, we make a step further from [5]. We study the K -user Gaussian MAC with K independent additive Gaussian interferences each of which known to exactly one transmitter non-causally, which is an extension of the two-user doubly-dirty MAC. With the help of a binary expansion model of the original problem, we propose a layered modulo-lattice scheme that realizes distributed interference cancellation, and characterize the capacity region to within a constant gap, for arbitrary channel parameters. The binary expansion model uncovers the underlying layered structure of the original Gaussian problem, which leads naturally to the layered architecture and the converse proof.

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APPENDIX A
PROOF OF LEMMA 4.1

Let

$$y_k := \sum_{i=k}^K x_i + \sum_{i=k}^K s_i + z. \quad (63)$$

If $R_{[1:K]}$ is achievable, for any $k \in [1 : K]$ by Fano's inequality and data processing inequality, we have

$$N \left(\sum_{i=k}^K R_i - \epsilon_N \right) \quad (64)$$

$$\leq I(w_{[k:K]}; y^N | w_{[1:k-1]}) \quad (65)$$

$$\stackrel{(a)}{\leq} I(w_{[k:K]}; y^N | w_{[1:k-1]}, s_{[1:k-1]}^N) \quad (66)$$

$$= h(y^N | w_{[1:k-1]}, s_{[1:k-1]}^N) - h(y^N | w_{[1:K]}, s_{[1:k-1]}^N) \quad (67)$$

$$= h(y_k^N | w_{[1:k-1]}, s_{[1:k-1]}^N) - h(y_k^N | w_{[1:K]}, s_{[1:k-1]}^N) \quad (68)$$

$$\stackrel{(b)}{=} I(w_{[k:K]}; y_k^N) = I(w_{[k:K]}, s_{[k:K]}^N; y_k^N) - I(s_{[k:K]}^N; y_k^N | w_{[k:K]}) \quad (69)$$

$$\stackrel{(c)}{=} h(y_k^N) - h(z^N) - \sum_{i=k}^K h(s_i^N) + h(s_{[k:K]}^N | y_k^N, w_{[k:K]}) \quad (70)$$

$$= h(y_k^N) - h(z^N) - \sum_{i=k}^K h(s_i^N) + \sum_{i=k}^K h(s_i^N | y_k^N, w_{[k:K]}, s_{[k:i-1]}^N) \quad (71)$$

$$\stackrel{(d)}{\leq} -h(z^N) + h(y_k^N) - h(s_k^N) + h(s_k^N | y_k^N) - \sum_{i=k+1}^K h(s_i^N) + \sum_{i=k+1}^K h(s_i^N | y_i^N, w_{[i:K]}) \quad (72)$$

$$\stackrel{(e)}{\leq} -h(z^N) + h(y_k^N | s_k^N) - \sum_{i=k+1}^K h(s_i^N) \quad (73)$$

$$+ \sum_{i=k+1}^K \min \left\{ h(s_i^N), h \left(x_i^N + \sum_{l=i+1}^K (x_l^N + s_l^N) + z^N \right) \right\} \quad (74)$$

$$\stackrel{(f)}{\leq} N \bar{R}_k (\text{SNR}_{[k:K]}, \text{INR}_{[k+1:K]}; K), \quad (75)$$

where $\epsilon_N \rightarrow 0$ as $N \rightarrow \infty$. (a) is due to the facts that conditioning reduces entropy and that $s_{[1:k-1]}^N$ is independent of $w_{[k:K]}$. (b) is due to the fact that $(w_{[k:K]}, s_{[k:K]}^N, y_k^N)$ and $(w_{[1:k-1]}, s_{[1:k-1]}^N)$ are independent. (c) is due to the fact that $\{w_{[k:K]}, s_{[k:K]}^N\}$ are mutually independent and y_k^N

is a function of $\left\{w_{[k:K]}, s_{[k:K]}^N\right\}$. (d) is due to conditioning reduces entropy and the fact that $\left(w_{[i:K]}, s_{[i:K]}^N, y_i^N\right)$ and $\left(w_{[k:i-1]}, s_{[k:i-1]}^N\right)$ are independent. (e) is due to the fact that $y_i^N = x_i^N + s_i^N + \sum_{l=i+1}^K (x_l^N + s_l^N) + z^N$. Finally, (f) is due to the fact that

$$h\left(y_k^N | s_k^N\right) = h\left(x_k^N + \sum_{i=k+1}^K (x_i^N + s_i^N) + z^N \middle| s_k^N\right) \leq h\left(x_k^N + \sum_{i=k+1}^K (x_i^N + s_i^N) + z^N\right),$$

Gaussian distribution maximizes the unconditional entropy, and $\text{Var}\left[x_i^N + s_i^N\right] \leq 2\text{Var}\left[x_i^N\right] + 2\text{Var}\left[s_i^N\right]$ for any i . Proof complete.

APPENDIX B

PROOF OF THEOREM 4.3

We shall evaluate and upper bound the gap

$$\delta_k := \bar{R}_k\left(\text{SNR}_{[k:K]}, \text{INR}_{[k+1:K]}; K\right) - \underline{R}_k\left(\text{SNR}_{[k:K]}, \text{INR}_{[k+1:K]}; K\right). \quad (76)$$

For notational convenience, we denote $\Upsilon_i := \sum_{j=i}^K (\text{SNR}_j + \text{INR}_j)$.

First note that $\underline{R}_k\left(\text{SNR}_{[k:K]}, \text{INR}_{[k+1:K]}; K\right)$ can be lower bounded by

$$\underline{R}_k\left(\text{SNR}_{[k:K]}, \text{INR}_{[k+1:K]}; K\right) \quad (77)$$

$$\geq \frac{1}{2} \log(1 + k\text{SNR}_k + \Upsilon_{k+1}) - \sum_{i=k}^K \frac{1}{2} \log i - \sum_{i=k+1}^K \frac{1}{2} \log \left(\frac{1 + i\text{SNR}_i + \text{INR}_i + \Upsilon_{i+1}}{1 + i\text{SNR}_i + \Upsilon_{i+1}} \right) \quad (78)$$

$$\geq \frac{1}{2} \log(1 + \text{SNR}_k + \Upsilon_{k+1}) - \sum_{i=k}^K \frac{1}{2} \log i - \sum_{i=k+1}^K \frac{1}{2} \log \left(\frac{1 + i\text{SNR}_i + \text{INR}_i + \Upsilon_{i+1}}{1 + i\text{SNR}_i + \Upsilon_{i+1}} \right). \quad (79)$$

Also,

$$\bar{R}_k\left(\text{SNR}_{[k:K]}, \text{INR}_{[k+1:K]}; K\right) \quad (80)$$

$$= \frac{1}{2} \log(1 + 2\Upsilon_{k+1} + \text{SNR}_k) - \sum_{i=k+1}^K \frac{1}{2} \log^+ \left(\frac{\text{INR}_i}{1 + 2\Upsilon_{i+1} + \text{SNR}_i} \right) \quad (81)$$

$$\leq \frac{1}{2} \log(1 + \Upsilon_{k+1} + \text{SNR}_k) - \sum_{i=k+1}^K \frac{1}{2} \log^+ \left(\frac{\text{INR}_i}{1 + \Upsilon_{i+1} + \text{SNR}_i} \right) + \frac{1}{2}(K - k + 1). \quad (82)$$

Hence,

$$\delta_k \leq \sum_{i=k+1}^K \frac{1}{2} \log i + \sum_{i=k+1}^K \frac{1}{2} \log \left(\frac{1 + \Upsilon_{i+1} + i\text{SNR}_i + \text{INR}_i}{1 + \Upsilon_{i+1} + i\text{SNR}_i} \right) \quad (83)$$

$$- \sum_{i=k+1}^K \frac{1}{2} \log^+ \left(\frac{\text{INR}_i}{1 + \Upsilon_{i+1} + \text{SNR}_i} \right) + \frac{1}{2}(K - k + 1) \quad (84)$$

$$= \sum_{i=k}^K \frac{1}{2} \log i + \sum_{i=k+1}^K (\zeta_i - \xi_i) + \frac{1}{2}(K - k + 1), \quad (85)$$

where $\zeta_i := \frac{1}{2} \log \left(\frac{1 + \Upsilon_{i+1} + i\text{SNR}_i + \text{INR}_i}{1 + \Upsilon_{i+1} + i\text{SNR}_i} \right)$ and $\xi_i := \frac{1}{2} \log^+ \left(\frac{\text{INR}_i}{1 + \Upsilon_{i+1} + \text{SNR}_i} \right)$.

- 1) If $\text{INR}_i \leq 1 + \Upsilon_{i+1} + \text{SNR}_i$, then $\xi_i = 0$, and $\zeta_i \leq \frac{1}{2} \log(1 + 1) = \frac{1}{2}$.
- 2) If $\text{INR}_i > 1 + \Upsilon_{i+1} + \text{SNR}_i$, then

$$\zeta_i - \xi_i = \frac{1}{2} \log \left(\frac{(1 + \Upsilon_{i+1} + i\text{SNR}_i + \text{INR}_i)(1 + \Upsilon_{i+1} + \text{SNR}_i)}{\text{INR}_i(1 + \Upsilon_{i+1} + i\text{SNR}_i)} \right) \quad (86)$$

$$\leq \frac{1}{2} \log \left(\frac{1 + \Upsilon_{i+1} + i\text{SNR}_i + \text{INR}_i}{\text{INR}_i} \right) \quad (87)$$

$$\leq \frac{1}{2} \log(i + 1). \quad (88)$$

Therefore, combining 1) and 2), for all $i \in [k + 1 : K]$, $\zeta_i - \xi_i \leq \frac{1}{2} \log(i + 1)$. Hence,

$$\delta_k \leq \sum_{i=k}^K \frac{1}{2} \log i + \sum_{i=k+1}^K \frac{1}{2} \log(i + 1) + \frac{1}{2}(K - k + 1) \quad (89)$$

$$\leq \sum_{i=k}^K \frac{1}{2} \log K + \sum_{i=k+1}^{K-1} \frac{1}{2} \log K + \frac{1}{2} \log(K + 1) + \frac{1}{2}(K - k + 1) \quad (90)$$

$$\leq (K - k + 1) \log K + \frac{1}{2}(K - k + 1), \quad (91)$$

since $K + 1 \leq K^2$ for $K \geq 2$.